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SPARSE GRAPH SIGNAL RECONSTRUCTION ON CIRCULANT GRAPHS WITH PERTURBATIONS

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ABSTRACT

In this work, we consider the problem of K-sparse graph signal recovery on perturbed circulant graphs, simulating network clusters within a large network, as an extension of the framework of sampling and reconstructing signals with a finite rate of innovation (FRI) in the classical domain to the graph domain. In light of the fact that the GFT-basis on a circulant graph $G = (V, E)$ is, up to a permutation, given by the columns of the DFT-matrix, we develop a reconstruction approach, whereby we employ a variation of Prony's method, along with further iterative denoising involving Cadzow's algorithm, to recover the sparse graph signal represented on the vertices of G. In particular, given a dimensionality-reduced approximation of the GFT of a K-sparse graph signal $\mathbf{x} \in \mathbb{R}^N$ lying on the vertices of a perturbed circulant graph G, defined as $\mathbf{y} = \mathbf{C}\mathbf{U}^H\mathbf{x}$, where $\mathbf{C}$ and $\mathbf{U}$ represent an appropriate coefficient matrix and the partial GFT-basis, respectively, we can perfectly reconstruct $\mathbf{x}$. We give preliminary results and discuss generalizations to arbitrary graphs.

***Index Terms*—** finite rate of innovation, circulant graph, graph signal processing

1. INTRODUCTION

The field of graph signal processing has sought to provide an extension, and possible alternative, to traditional signal processing, by establishing comparable concepts, properties, and operations in the graph domain, motivated by the appealing potential of graphs to capture complexity beyond the classical domain when used for the representation and processing of large data sets. Hereby, prior research has focused on developing a wide range of equivalencies, such as basic operations on graph signals ([2],[3]), graph-dependent downsampling strategies ([2],[3], [4]) as well as graph wavelets ([5], [6]) and filterbanks ([7], [8], [9]), in addition to further analytic explorations of the smoothness and compressibility of graph signals with respect to the given graph ([10], [11]).
The class of circulant graphs, has been particularly noted for its set of properties, which facilitate downsampling and shifting operations ([3], [9], [12]). In addition, the graph spectral representation, or Graph Fourier Transform (GFT), on undirected, circulant graphs, is, up to a permutation, given by the DFT-matrix. This has inspired our investigation of K-sparse graph signal recovery on network clusters as an extension to FRI-sampling and reconstruction of sparse signals in the classical domain. Given a large network, we propose to model individual network clusters as (un-)weighted and undirected circulant subgraphs, which are linked via inter-connecting edges on a main graph $G = (V, E)$, and are subject to perturbations in form of the addition and/or removal of randomly chosen edges. We present a variation of Prony's method [13] in a novel model of blockwise reconstruction operations with dimensionality reduction, whereby we operate on each subgraph individually. We give preliminary results and propose the nearest circulant approximation scheme ([14], [15]) for generalizations to arbitrary graphs. This paper is organized as follows: we begin with a brief summary of the preliminaries in Section 2, following which we enlarge upon our novel framework for the reconstruction of sparse graph signals on (perturbed) circulant graphs in Section 3. Section 4 contains preliminary experimental results for the developed method, and in Section 5 we make concluding remarks.

This work will be in part presented at the GlobalSIP conference, '14 [1].

2. PRELIMINARIES

We provide a short overview of basic results, which we will draw on in the main body of this work.

2.1. FRI-Signal Sampling and Reconstruction in the Classical Domain

The class of FRI-signals, comprising non-bandlimited signals with a finite rate of innovation, can be sampled and perfectly reconstructed using kernels of compact support, which satisfy certain Strang-Fix conditions, and a local reconstruction algorithm (Prony's method) [13]. In the discrete time domain, consider a signal $\mathbf{x} \in \mathbb{R}^N$ with $||\mathbf{x}||_0 = K$, and define the measurement vector $\mathbf{y}$ in the Fourier domain, such that $\mathbf{y} = \mathbf{F}\mathbf{x}$, where $\mathbf{F} \in \mathbb{C}^{N \times N}$ is the DFT-matrix. Then the signal samples y_n are given by

$$y_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{K-1} x_{c_k} e^{-i2\pi c_k n/N} = \sum_{k=0}^{K-1} \alpha_k u_k^n \qquad (1)$$

where x_{c_k} is the weight of $\mathbf{x}$ at index c_k, and $\alpha_k = x_{c_k}/\sqrt{N}$ and $u_k = e^{-i2\pi c_k n/N}$ represent the amplitudes and locations respectively. We can then perfectly reconstruct the vector $\mathbf{x}$ based on $M = 2K$ consecutive sample values of $\mathbf{y}$ using Prony's method. If the signal at hand contains noise, such as in form of additive Gaussian noise $\mathbf{n}$, giving $\hat{\mathbf{y}} = \mathbf{y} + \mathbf{n}$, we require a larger number of samples $M \geq 2K$ as well as need to apply denoising schemes, such as Cadzow's algorithm [17], to achieve perfect reconstruction.

2.2. Graph Signal Processing

In this paper, we consider a graph $G = (V, E)$, defined by a vertex set V, $|V| = N$, and an edge set E, which is undirected, connected and (un-)weighted, without self-loops. The graph G is characterized by its adjacency matrix $\mathbf{A}$, with $A_{i,j} > 0$ if there is an edge between nodes i and j, and $A_{i,j} = 0$ otherwise, and its degree matrix $\mathbf{D}$, which is diagonal with entries $D_{i,i} = \sum_j A_{i,j}$. The non-normalized graph Laplacian matrix $\mathbf{L} = \mathbf{D} - \mathbf{A}$ of G, has a complete set of orthonormal eigenvectors $\{\mathbf{u}_l\}_{l=0}^{N-1}$, with corresponding nonnegative eigenvalues $\{\lambda_l\}_{l=0}^{N-1}$, specified in increasing order $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_{N-1}$.

A graph signal $\mathbf{x}$ is a real-valued scalar function defined on the vertices of a graph G, with sample value $x(i)$ at node i, and can be represented as the vector $\mathbf{x} \in \mathbb{R}^N$ [2]. Analogously to the classical domain, the Graph Fourier Transform (GFT) $\mathbf{X}^G$ of a graph signal $\mathbf{x}$ defined on G, is the representation in terms of the graph Laplacian eigenbasis $\mathbf{U} = [\mathbf{u}_0|\cdots|\mathbf{u}_{N-1}]$: $\mathbf{X}^G = \mathbf{U}^H\mathbf{x}$, where H denotes the Hermitian transpose [2].

A circulant graph G is characterized by a generating set $S = \{s_1, \ldots, s_M\}$, with $0 < s_k \leq N-1$, such that there exists an edge between nodes $(i, (i+s_k)_N)$, for every $s_k \in S$; alternatively, any graph defined through a circulant graph Laplacian matrix is circulant [3].

2.3. Circulant Matrix Theory

In the general setting of a non-circulant, arbitrary graph, we need to establish a means to detect the nearest circulant structure as determined by a given error norm. In particular, the Chan circulant matrix ([14], [15]) gives the nearest circulant matrix $\mathbf{C} \in \mathbb{R}^{N\times N}$ to a given adjacency matrix $\mathbf{A} \in \mathbb{R}^{N\times N}$ in Frobenius norm $||\mathbf{A} - \mathbf{C}||_F$, by averaging over diagonals. Let $\mathbf{\Pi}$ be a circulant matrix with first row $\pi = [0\ 1\ 0\ \ldots\ 0]$; the nearest circulant matrix $\mathbf{C}$, with first row $\mathbf{c} = [c_0\ c_1\ \ldots\ c_{N-1}]$, to a given $\mathbf{A}$ is then determined by the following Frobenius inner product:

$$c_k := \frac{1}{N}\langle \mathbf{A}, \mathbf{\Pi}^k\rangle_F = tr(\mathbf{A}^T\mathbf{\Pi}^k), \quad k = 0, \ldots, N-1 \tag{2}$$

The GFT-basis on circulant graphs can be expressed as a permutation of the columns of the DFT-matrix due to the circulant structure of the graph Laplacian [12], however, as a result of the occurring eigenvalue multiplicities, it is not unique. In general, a circulant matrix $\mathbf{C}$, with first row $\mathbf{c} = [c_0\ c_1\ \ldots\ c_{N-1}]$, is diagonalizable by the DFT-matrix, with the following resulting spectrum of corresponding eigenvalues λ_j:

$$\lambda_j = c_0 + c_1\omega_j + c_2\omega_j^2 + \cdots + c_{N-1}\omega_j^{N-1}, \tag{3}$$
$$\text{where} \quad \omega_j = e^{-\frac{i2\pi j}{N}},\ j = 0, ..., N-1.$$

The eigenvalue multiplicity distribution of a circulant graph G with an arbitrary generating set S, is not generally defined, and can only be determined by applying the exhaustive search approach; however one can infer the following basic result on the occurrence of odd and even multiplicities of eigenvalues of circulant matrices [18]:

Definition 1: Let $\mathbf{B} \in \mathbb{R}^{n\times n}$ be a real, symmetric, circulant matrix. Then, $\mathbf{B}$ has n orthogonal eigenvectors and n eigenvalues. If n is odd, there is one eigenvalue with odd multiplicity, and if n is even, there are either two eigenvalues or none with odd multiplicity. All other eigenvalues have even multiplicity.

As an example, we consider the eigenvalue distribution of the simple cycle, which is defined as: $\lambda_k = 2 - 2\cos\left(\frac{2\pi k}{N}\right), \quad k = 0, \cdots, N-1$. All eigenvalues occur with multiplicity $m_i = 2$, with the exception of $\lambda_0 = 0$ (and $\lambda_N = 4$ if N is even) with $m_0 = 1$.

3. SPARSE GRAPH SIGNAL RECONSTRUCTION

Inspired by the framework of sampling and reconstructing sparse signals within the FRI framework in the classical domain, we wish to consider extensions to the graph domain by sampling and reconstructing sparse graph signals on the vertices of network clusters, representing groups of strongly connected entities, within large networks. In our proposed approach, we model the aforementioned clusters as undirected, (un-)weighted circulant subgraphs, which are connected through few edges on a large main graph $G = (V, E)$, and subjected to perturbations in form of the addition and/or removal of randomly generated edges, simulating randomness and complexity of realistic clustered networks. Hereby, we operate on each subgraph individually by applying graph cuts [16] on inter-connecting edges between the perturbed circulant subgraphs beforehand.

Let $\mathbf{x} \in \mathbb{R}^N$, with $||\mathbf{x}||_0 = K$, denote a sparse graph signal defined on the vertices of a graph G, and its representation in the Graph Fourier (GFT) domain be $\hat{\mathbf{y}} = \mathbf{U}^H\mathbf{x}$, where $\mathbf{U} = [\mathbf{u}_0|\ldots|\mathbf{u}_{N-1}] \in \mathbb{R}^{N\times N}$ is the corresponding matrix of eigenvectors. Given that a permutation of the DFT-matrix $\mathbf{F} \in \mathbb{C}^{N\times N}$ is a possible choice for the basis $\mathbf{U}$ when G is circulant, we introduce a coefficient matrix $\mathbf{C} \in \mathbb{C}^{N\times N}$, resulting in the new measurement vector $\mathbf{y} = \mathbf{C}\hat{\mathbf{y}} = \mathbf{C}\mathbf{U}^H\mathbf{x}$, whereby $\mathbf{F} = \mathbf{C}\mathbf{U}^H$. Depending on the eigenvalue multiplicity distribution of the graph at hand, we may need to impose a further permutation $\sigma_{\tilde{\Lambda}_j}$ on the basis $\mathbf{U}$ beforehand, according to the eigenvalue sequence $\tilde{\Lambda}_j$ obtained by taking the DFT of the first row of the graph Laplacian of the circulant G (or its nearest circulant approximation), and arranging eigenvalues of the same subspace j together. In the case of dimensionality reduction, this allows to obtain the best least-squares approximation of the matrix $\mathbf{C}$.

In accordance with the classical domain, we can then perfectly recover $\mathbf{x}$ from $2K$ consecutive samples of $\mathbf{y}$ using Prony's method. When the circulant graph at hand is perturbed, we apply denoising schemes, such as Cadzow's algorithm, to the given samples prior to reconstruction as well as may need to employ further iterative denoising based on removal of the perturbation matrix $\mathbf{E} = \mathbf{C}\mathbf{U}^H - \mathbf{F}$ at the current estimate $\hat{\mathbf{x}}$ from the measurement vector $\mathbf{y}$ at iteration i: $\mathbf{y}^{i+1} = \mathbf{y} - \mathbf{E}\hat{\mathbf{x}}^i$. For a sufficiently large number of given samples M, and number of iterations i, we can achieve perfect reconstruction.

Overall, we formulate the FRI-framework on graphs as follows: We can perfectly reconstruct a sparse graph signal $\mathbf{x} \in \mathbb{R}^N$ on the vertices of a graph $G = (V, E)$, given a dimensionality-reduced approximation of the GFT of $\mathbf{x}$, defined through the measurement vector $\mathbf{y} = \mathbf{C}\mathbf{U}_P^H\mathbf{x}$, $\mathbf{y} \in \mathbb{C}^M$, where $\mathbf{U}_P$ represents a subset of P eigenvectors of the GFT-basis $\mathbf{U}$ and $\mathbf{C} \in \mathbb{C}^{M\times P}$ denotes an appropriate coefficient matrix, with suitably chosen $M < P < N$. Hereby, the aforementioned matrix product is designed to approximate the first M rows of the DFT-matrix $\mathbf{F}_M \approx \mathbf{C}\mathbf{U}_P^H$.

Furthermore, we impose a dimensionality offset $P = f(M)$, requiring the coefficient matrix $\mathbf{C}$ to be fat, due to the occurring eigenvalue multiplicities of the graph Laplacian of a circulant graph. In the case of a simple cycle, for instance, we can perfectly recover $\mathbf{x}$ with only $P = 2M - 1 = 4K - 1$ GFT-sample values, since the eigenvalues occur with a maximum multiplicity of $m = 2$ (except for $\lambda_0 = 0$, and $\lambda_N = 4$ if N is even, with $m = 1$).

When the perturbation on the graph at hand is more invasive relative to its structure, we observe the occurrence of eigensubspace-swap phenomena, whereby $\tilde{\lambda}_i = \lambda_{i\pm1}$ for perturbed and unperturbed ordered eigenvalues $\tilde{\lambda}_i$ and λ_i at the ith-position, respectively. This may be corrected for through additional permutation schemes or alternative circulant approximation schemes, which are subject to further investigation.

We summarize our proposed approaches in Algorithm 1, whereby we note that Option 2 applies to a subset of circulant graphs with certain generating sets; circulant graphs in the complementary set have exhibited destructive behavior in the reconstruction process due to highly localized perturbation noise.

Algorithm 1 Sparse Graph Signal Recovery on Network Clusters

1: **Input**: The clustered graph G with adjacency matrix $\mathbf{A}$, and graph signal $\mathbf{x}$, with $||\mathbf{x}||_0 = K$, on the vertices of G

2: Decompose G into T disconnected subgraphs $\{G_l\}_{l=1}^{T}$ using e.g. the graph cut method. Map $\mathbf{x}$ appropriately to the disconnected subgraphs, resulting in T signals $\{\mathbf{x}_l\}_{l=1}^{T}$ with sparsity $\{K_l\}_{l=1}^{T}$. Apply the following scheme(s) on each subgraph individually

3: **Option 1**: Compute the nearest circulant $\tilde{\mathbf{A}}_l$ to $\mathbf{A}_l$, via (2). Map $\mathbf{x}_l$ to the vertices of $\tilde{G}_l$, and impose a permutation $\sigma_{\tilde{\Lambda}_j}$ on $\tilde{\mathbf{U}}$ of $\tilde{G}_l$, as required. Compute $\mathbf{C}$ via LS: $\mathbf{C}^T = (\tilde{\mathbf{U}}_P^H)^T \backslash \mathbf{F}_M^T$. Only $P = 4K_l - 1$ samples are required for perfect reconstruction, at best. Ensure that $\tilde{G}_l$ has the required minimum of multiplicities (or adjust P accordingly). Store GFT-vector $\hat{\mathbf{y}} = \tilde{\mathbf{U}}_P^H \mathbf{x}_l \in \mathbb{C}^P$

4: **Option 2**: Model the graph as circulant with a perturbation, by determining the nearest circulant $\tilde{\mathbf{A}}_l$ to the given $\mathbf{A}_l$, and imposing the permutation $\sigma_{\tilde{\Lambda}_j}$ on $\mathbf{U}$ of G_l, if required. Compute $\mathbf{C}$, and create $\mathbf{y} = \mathbf{C}\mathbf{U}_P^H \mathbf{x}_l$ for suitable M, and $P = f(M)$ (where the mapping f depends on the multiplicities of $\tilde{G}_l$). Apply the proposed scheme at $P \geq 4K_l - 1 \geq f(M)$ samples:

5: Denoise $\mathbf{y}$ with Cadzow's algorithm, and recover $\hat{\mathbf{x}}_l$ through Prony's method

6: Do further iterative denoising, $\mathbf{y}^{i+1} = \mathbf{y} - \mathbf{E}\hat{\mathbf{x}}_l^i$, with $\mathbf{E} = \mathbf{C}\mathbf{U}_P^H - \mathbf{F}_M$, as required, while repeating 5. Store GFT-vector $\hat{\mathbf{y}} = \mathbf{U}_P^H \mathbf{x}_l \in \mathbb{C}^P$.

4. EXPERIMENTAL RESULTS

We subject a simple cycle graph to perturbations in form of two additional, randomly distributed edges, and map a set of 100 randomly generated sparse graph signals with i.i.d. entries to its vertices; we apply the proposed recovery (with denoising) scheme on the corresponding measurement vectors (see Option 2 in Algorithm 1). Figure 1 shows the average reconstruction performance per iteration in form of the average location error between the estimated and true entry locations of the randomly generated sparse vector $\mathbf{x}$ for 100 trials, given the dimensionality reduced measurement vector $\mathbf{y} \in \mathbb{C}^M$, and $P = 2M - 1$ GFT-samples.

5. CONCLUSION AND FUTURE WORK

The developed FRI-framework for sparse graph signal sampling and reconstruction on perturbed circulant graphs represents a promising new venture in graph signal processing, which facilitates recovery with dimensionality reduction in the graph domain. In the future, we wish to further investigate and develop the specific graph-theoretic interpretation of this approach, and in particular, of the employed nearest circulant approximation scheme and possibly suitable alternatives. Furthermore, we wish to gain a deeper understanding of the observed perturbation phenomena as well as obtain a well-defined description of the set of circulant graphs satisfying our scheme.

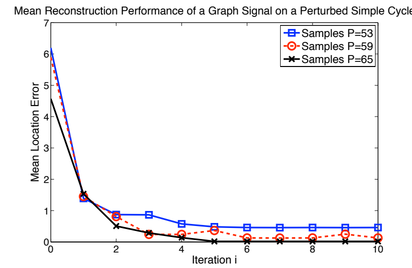

Fig. 1. Reconstruction Performance on a Perturbed Simple Cycle ($N = 256$), for 100 randomly generated sparse signals $\mathbf{x}_l$ ($K = 4$, minimum separation of 3 between entries)

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